# Towards a Rule-Based Interpretation of Conditional Defaults

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## Abstract

For nonmonotonic reasoning, a default conditional $\alpha \rightarrow \beta$ has most often been informally interpreted as a defeasible version of a classical conditional, usually the material conditional. There is however an alternative interpretation, in which a default is regarded essentially as a rule, leading from premises to conclusion. In this paper, we present a family of logics, based on this alternative interpretation. A general semantic framework under this "rule-based" interpretation is developed, and associated proof theories for a family of weak conditional logics is specified. Nonmonotonic inference is easily defined in these logics. One obtains a rich set of nonmonotonic inferences concerning the incorporation of irrelevant properties and of property inheritance. Interestingly, the logics presented here are weaker than the commonly-accepted base conditional approach for defeasible reasoning. However, this interpretation resolves problems that have been associated with previous approaches.

## 1 Introduction

A major approach in nonmonotonic reasoning is to represent a default as an object that one can reason about, either as a conditional in some object language, or as a nonmonotonic consequence operator. Thus for example "an adult is (typically or normally) employed" might be represented $a \rightarrow e$, where $\rightarrow$ represents a default or normality conditional, distinct from the material conditional $\supset$. In such approaches, one can typically derive other defaults from a given set of defaults. There has been widespread agreement concerning just what principles should constitute a minimal logic. Such a minimal logic would form a "conservative core" of defaults common (so it is suggested) to all approaches to nonmonotonic reasoning. However, the resulting default conditional is quite weak, at least compared with the material conditional, in that it does not (in fact, *should not*) fully support principles such as strengthening of the antecedent, transitivity, and modus ponens.

Since one would want to obtain these latter properties by default, such logics are extended nonmonotonically by a "closure" operation or step. This closure operation has been defined, for example, in terms of a preferred subset of the models of a theory. In the resulting set of models, one obtains strengthening of the antecedent, transitivity, or (effectively) modus ponens, wherever feasible. Essentially then, there are two components to default reasoning within such a system. First, there is a standard, monotonic logic of conditionals that expresses relations among defaults that are deemed to always hold. Second, there is a nonmonotonic mechanism for obtaining defaults (and default consequences) where justified. In essence, these approaches treat the default conditional like its classical counterpart, the material conditional, where feasible or by default.

While this work captures an important notion of default entailment – perhaps *the* most important notion – it is not without difficulties. As described in the next section, some principles of the suggested core logic are not uncontentious; as well, there are examples of default reasoning in which one obtains undesirable results. Lastly, there are more recent approaches, notably addressing causality, in which one requires a weaker notion of default inference, rejecting, for example, contrapositive default inferences. In response to these points, I suggest that there is a second, distinct, interpretation of default conditionals, in which a default is regarded more like a rule, with properties more in line with a rule of inference than a weakened classical conditional.

In the following sections I describe an approach under this second interpretation. I begin by proposing an exceptionally weak logic of conditionals; from this basis a family of conditional logics is defined. Given a default conditional $\alpha \rightarrow \beta$, the underlying intuition is that $\alpha$ supplies a context in which, all other things being equal, $\beta$ normally holds or, more precisely, in the context of $\alpha$, the proposition expressed by $\alpha \wedge \beta$ is more "normal" than that expressed by $\alpha \wedge \neg\beta$, which is written as $\|\alpha\|_w \cap \|\neg\beta\|_w < \|\alpha\|_w \cap \|\beta\|_w$ or equivalently $\|\alpha \wedge \neg\beta\|_w < \|\alpha \wedge \beta\|_w$. Notably, all of the logics that are considered are weaker than the aforementioned "conservative core". It proves to be the case however that a nonmonotonic operation is very easily defined; this nonmonotonic step essentially specifies that a property is irrelevant with respect to a default unless it is known to be relevant. Thus, given a default $\alpha \to \beta$, one would want to also accept the strengthening $(\alpha \wedge \gamma) \to \beta$ whenever "reasonable". The nonmonotonic step corresponds to formalising the conclusion that that $\|\alpha \wedge \gamma\|_w \cap \|\neg\beta\|_w < \|\alpha \wedge \gamma\|_w \cap \|\beta\|_w$, given that $\|\alpha\|_w \cap \|\neg\beta\|_w < \|\alpha\|_w \cap \|\beta\|_w$. This nonmonotonic step easily admits inferences that in other approaches has required significant formal machinery. As well, I show that the aforementioned difficulties that arise in interpreting a default as a weak classical conditional do not arise here.

This distinction between treating a default as a conditional or as a rule is not new. However a logic (that is, with both semantics and proof theory) capturing this interpretation has not been investigated previously, nor has a fully general nonmonotonic closure operator been developed under this interpretation.

The next section reviews previous work in conditional approaches to nonmonotonic reasoning. Section 3 informally reviews the approach while Section 4 describes a family of weak conditional logics. Section 5 considers the incorporation of a nonmonotonic extension to a conditional knowledge base. Section 6 is a discussion.

# 2 Conditional Logics and Nonmonotonic Reasoning

## 2.1 Conditional Logics

In recent years, much attention has been paid to conditional approaches to default reasoning. Such approaches address defeasible conditionals whose meaning is based on notions of preference among worlds or interpretations. Thus, the default that a bird normally flies can be represented propositionally as $b \to f$.[1] These approaches are typically expressed using a modal logic in which the connective $\to$ is a binary modal operator. The intended meaning of $\alpha \to \beta$ is approximately "in the least worlds (or most preferred worlds) in which $\alpha$ is true, $\beta$ is also true". Possible worlds (or, again, interpretations) are arranged in at least a partial preorder, reflecting a metric of normality or preferredness on the worlds. Given a set of defaults $\Gamma$, default entailment with respect to $\Gamma$, $\mid\!\sim_\Gamma$, can be defined via:

$$\text{If} \quad \Gamma \vdash \alpha \to \beta \quad \text{then} \quad \alpha \mid\!\sim_\Gamma \beta. \tag{1}$$

There has been a remarkable convergence or agreement on what inferences ought to be common to all nonmonotonic systems, and in the literature a seeming diversity of conditional approaches essentially allows the same inferences. These include approaches based on intuitions from probability theory such as $\epsilon$-entailment [Pearl, 1988] (or 0-entailment or $p$-entailment [Adams, 1975]), from qualitative possibilistic logic [Dubois *et al.*, 1994], as well as modal-logic based approaches such as preferential entailment [Kraus *et al.*, 1990], C4 [Lamarre, 1991], and CT4 [Boutilier, 1994], Consequently it has been suggested that the resulting set of inferences may be taken as specifying a *conservative core* [Pearl, 1989] that arguably should be common to all default inference systems. One expression of this logic of conditionals is as follows. The logic includes classical propositional logic and the following rules and axioms:[2]

**RCEA/LLE:** From $\vdash \alpha \equiv \beta$ infer $\vdash (\alpha \to \gamma) \equiv (\beta \to \gamma)$.

**RCM/RW:** From $\vdash \beta \supset \gamma$ infer $\vdash (\alpha \to \beta) \supset (\alpha \to \gamma)$

**ID/Ref:** $\alpha \to \alpha$

**CC/And:** $((\alpha \to \beta) \wedge (\alpha \to \gamma)) \supset (\alpha \to \beta \wedge \gamma)$

**RT/Cut:** $((\alpha \to \beta) \wedge (\alpha \wedge \beta \to \gamma)) \supset (\alpha \to \gamma)$

**ASC/CM:** $((\alpha \to \beta) \wedge (\alpha \to \gamma)) \supset (\alpha \wedge \beta \to \gamma)$

**CA/Or:** $((\alpha \to \gamma) \wedge (\beta \to \gamma)) \supset (\alpha \vee \beta \to \gamma)$

Following [Lamarre, 1991] we call the above logic $C4$, as the conditional logic based on a S4-like accessibility relation (although Lamarre's axiomatisation is not exactly as given above). Note however, these principles are not uncontentious; for example, [Poole, 1991] can be viewed as arguing against **CC/And**. Likewise, [Neufeld, 1989] suggests against **CA** in some cases.

The semantics of these approaches is generally phrased in terms of a modal framework, in which possible worlds are ranked by a notion of relative normality or unexceptionalness. The underlying modal logic is generally taken to be S4 in which accessibility between worlds is given by a reflexive, transitive binary relation. A conditional $\alpha \to \beta$ is true at a world just when there is an accessible world in which $\alpha \wedge \beta$ is true and $\alpha \supset \beta$ is true at all worlds that are less or equally exceptional, or if there are no accessible $\alpha$ worlds. Thus, "birds fly", $b \to f$, is true if, in the least $b$-worlds, $b \supset f$ is true. Since penguins are birds (either $\Box(p \supset b)$ or $p \to b$) but penguins don't fly ($p \to \neg f$), this means that the least exceptional world in which there are penguins are more exceptional than the least worlds in which there are birds.

The resulting logic is weak. For example, the following relations which hold for the material conditional do not hold for the weak conditional:

**Strengthening:** From $\alpha \to \gamma$ infer $\alpha \wedge \beta \to \gamma$.

**Transitivity:** From $\alpha \to \beta$ and $\alpha \to \gamma$ infer $\alpha \to \gamma$.

**Contraposition:** From $\alpha \to \gamma$ infer $\neg\gamma \to \neg\alpha$.

**Modus ponens:** From $\alpha \to \beta$ and $\alpha$ infer $\beta$.

[1] An alternative is to treat the conditional as a nonmonotonic inference operator, $b \mid\!\sim f$. In a certain sense these approaches can be considered equivalent [Boutilier, 1994]; here, for simplicity, we remain within the conditional logic framework.

[2] Two systems of nomenclature have arisen, one associated with conditional logic and one with nonmonotonic consequence operators. We list both (when both exist) when first presenting an axiom or rule.

Nor would we want these principles to always hold for defaults. On the other hand, it would seem that one would want these properties to hold by default. Thus given that birds normally fly, and we are presented with a green bird, we would like to conclude that it flies. Clearly, this is not something that can be done *within* the logic; that is, given that a bird is asserted to fly by default, one cannot thereby conclude via (1) that a green bird flies by default. That is, simply put, the inference $b \to f \vdash b \wedge g \to f$ does not obtain, or $\{b \to f,\ b \wedge g \to \neg f\}$ is satisfiable. The problem is that there is nothing requiring preferred worlds in which birds fly to include among them green-bird worlds.

## 2.2 Nonmonotonic Extensions to Conditional Logics

Given the above considerations, various means of strengthening the logic to incorporate strengthening or transitivity in a principled fashion have been proposed. We focus in this subsection on two well-known approaches for nonmonotonically extending, or taking the (conditional) closure of, a conditional knowledge base.

*Rational closure* [Lehmann and Magidor, 1992] is an exemplar of a set of approaches that assumes, in a semantic sense, that a world is as unexceptional as consistently possible.[3] Thus, given that birds fly, all other things being equal, a world where a bird flies will be ranked below one where it does not. Similarly, since there is no reason to suppose that greenness has any bearing on flight, one assumes that green-bird-flying worlds are ranked as low as possible. Hence one would expect to find that at the least green-bird worlds that $fly$ is true; similarly, at the least nongreen-bird worlds we would also expect to find that $fly$ is true. Hence green birds (normally) fly as do non-green birds. Define $\beta \prec \alpha$ by

$$\Diamond(\alpha \vee \beta) \wedge ((\alpha \vee \beta) \to \neg\alpha).$$

Thus, informally, at the least $\alpha \vee \beta$ worlds, $\neg\alpha$ is in fact true; hence $\beta$ is true at such worlds and any $\alpha$ world is not less than these worlds. From this we can define an ordering on formulas of classical logic. The sign $\vdash_{C4}$ stands for logical derivation in C4, as the representative of the systems discussed in the previous section.

**Definition 2.1** *Given a default theory $T$, the* degree *of a formula $\alpha$ is defined as follows:*

1. *$degree(\alpha) = 0$ iff for no $\delta$ do we have $T \vdash_{C4} \delta \prec \alpha$.*
2. *$degree(\alpha) = i$ iff $degree(\alpha)$ is not less than $i$ and $T \vdash_{C4} \beta \prec \alpha$ only if $degree(\beta) < i$*
3. *$degree(\alpha) = \infty$ iff $\alpha$ is assigned no degree in Parts 1 and 2 above.*

From this the closure operation is defined:

[3] As with the base logic of the previous subsection, there have been an number of other approaches, founded on different intuitions, but again converging to essentially the same system.

**Definition 2.2** *The* rational consequence *relation, with respect to default theory $T$ is given by:*

$$\alpha \mid\!\sim_T \beta \quad \textit{iff} \quad degree(\alpha) < degree(\alpha \wedge \neg\beta) \textit{ or } degree(\alpha) = \infty.$$

Consider the following example:

**Example 2.1**

$$T = \{b \to f,\ b \to w,\ p \to b,\ p \to \neg f\}.$$

Hence, birds fly and have wings, while penguins are birds that do not fly. We obtain inferences such as

$$b \wedge g \mid\!\sim_T f,\ \ b \wedge \neg g \mid\!\sim_T f,\ \ b \mid\!\sim_T \neg p,\ \text{ and }\ p \wedge g \mid\!\sim_T \neg f.$$

Notably, one does not obtain the result $p \mid\!\sim w$ even though this inference would appear to be sanctioned by the defaults $p \to b$ and $b \to w$. Thus, in the rational closure, one does not obtain inheritance of properties (in this case $w$) across exceptional subclasses ($p$). A further discussion of properties of the rational closure is deferred to the next subsection; however it is worth pointing out that the failure to allow full inheritance of properties has been addressed, for example in [Benferhat *et al.*, 1993] via the *lexicographic closure* of a set of defaults. However these extensions are syntax-dependent, and come at the expense of higher complexity than the original formulation.

A second, well-known approach is *conditional entailment* [Geffner and Pearl, 1992]. Conditional entailment was formulated in part to reconcile approaches exemplified by conditional logics on the one hand, and earlier approaches such as circumscription on the other. In conditional entailment, defaults are arranged in a partial order, determined in part by the specificity of a rule's antecedent. This priority order over the set of defaults $\Delta_{\mathcal{L}}$ is defined such that every set $\Delta$ of defaults in conflict with a default $r$ contains a default $r'$ that is less than that default in the ordering. Given this ordering on rules, an ordering on worlds can then be defined: If $\Delta(w)$ and $\Delta(w')$ are the defaults falsified by worlds $w$ and $w'$ respectively, then $w$ is preferred to $w'$ iff $\Delta(w) \neq \Delta(w')$, and for every rule in $\Delta(w) \setminus \Delta(w')$ there is a rule in $\Delta(w') \setminus \Delta(w)$ which has higher priority. As usual, $\beta$ is a default consequence of $\alpha$ just if $\beta$ is true in the most preferred $\alpha$ worlds. We obtain the same consequences given for Example 2.1 as for the rational closure; moreover we obtain that $p \mid\!\sim w$. However full inheritance of properties is not supported. Consider the following example [Geffner and Pearl, 1992]:

**Example 2.2**

$$T = \{b \to f,\ p \to s,\ s \to b,\ p \to \neg f\}.$$

(Thus, birds fly; penguins are shorebirds; shorebirds fly; but penguins don't fly.) The expected inference $p \mid\!\sim_T b$ does not obtain.

Rational closure and conditional entailment formalise important and interesting phenomena in nonmonotonic reasoning, and have found widespread application in the literature. However there are problems with both approaches when considered as a general approach to formalising reasoning with defaults or normality conditionals. Rational closure, for example, employs a very strong minimization criterion that is not always appropriate. Consider the following elaboration of an example given by John Horty:

**Example 2.3**

$$\top \to \neg f,\ b \to f,\ \top \to n,\ o \to \neg n.$$

(Normally one does not eat with the fingers ($f$), but one does when eating bread at a meal ($b$); normally one uses a napkin ($n$), but not when one is out of napkins ($o$).) The rational closure of these conditionals gives that, if one is not out of napkins ($\neg o$), one is not eating bread ($\neg b$). Clearly this interaction between unrelated defaults is undesirable.

In addition, consider the following example [Geffner and Pearl, 1992]:

**Example 2.4**

$$a \to e,\ u \to a,\ u \to \neg e,\ f \to a.$$

(That is, adults are normally employed, university students are normally adults but are not employed, and Frank Sinatra fans are normally adults.) In both conditional entailment and rational closure we obtain the default inference that Frank Sinatra fans are not university students. But this is a curious inference, since there is nothing in the example that would seem to relate Frank Sinatra fans to university students. So this is arguably an instance in which the result obtained is too strong. As well, if the conditional $u \to \neg e$ is dropped from the theory, one now loses the default inference that Frank Sinatra fans are not university students. In this instance, it seems strange that a nonmonotonic inference between Frank Sinatra fans and university students should be mediated by a person's being employed or not.

## 2.3 Reconsidering Defaults

As suggested, at least some of these examples do not necessarily reflect a problem with the approaches per se. Rather, our thesis is that there are (at least) two distinct interpretations that can be given to a default. First, there is the intuition that a default is essentially a weak version of the material conditional (or, in more recent approaches, necessary entailment), and should behave as such a conditional, except that it is defeasible. Note that conditional entailment explicitly adopts the intuition that a default is essentially a weak version of the material conditional. That is, the default $\alpha \to \beta$ is basically the same as $\top \to (\alpha \supset \beta)$ (i.e. the material counterpart normally holds) together with specificity information implicit in $\alpha$ [Geffner and Pearl, 1992, p. 232]. There are certainly instances (for example in diagnosing abnormalities in a circuit [Reiter, 1987]) where one wants, all other things being equal, a default to behave as a material conditional.

However, there are also situations where one does not want this behaviour. For example, consider the theory that asserts of a person that if they were to get a good evaluation at work, they would be happy. On the other hand, if they were to break their leg, they would not be happy:

**Example 2.5**

$$T = \{r \to h,\ bl \to \neg h\}.$$

In rational closure and conditional entailment, as well as in the corresponding circumscriptive abnormality theory, one obtains the inference $r \mid\!\sim \neg bl$: if someone gets a good evaluation then they won't break their leg. As well, it is not clear how such a theory could be repaired to avoid this conclusion; breaking the conflict by, for example adding $r \wedge bl \to \neg h$ doesn't solve the problem.

These considerations indicate that conditional closures, as represented by rational closure and conditional entailment (and by implication applying also to extensions of these works and related work) at times produce undesired conclusions. However, the monotonic consequences of the "conservative core" can also lead to unintuitive conclusions; consider the following example (of unknown source):

> A crime has been committed, of which the two suspects are John and Mary. In deciding who to arrest, the detective decides that if the murder weapon is found in John's room, then John will be arrested; if found in Mary's room then Mary will be arrested. If the weapon if found with John's fingerprints, then John will be arrested, and if Mary's then Mary.

We can symbolize this by:

**Example 2.6**

$$rJ \to J,\ \ rM \to M,\ \ fJ \to J,\ \ fM \to M,\ \ \Box\neg(J \wedge M).$$

What if the gun is found in John's room but with Mary's fingerprints (or vice versa)? Assume that to settle this conflict, it is decided that fingerprints decide the culprit. So we add

$$rJ \wedge fM \to M \quad \text{and} \quad rM \wedge fJ \to J.$$

With these defaults we can derive $rJ \to \neg fM$ contrary to one's intuitions.

These considerations indicate that there are situations in which nonmonotonic operations based on the default "core" logic lead to unintuitive results. Moreover, these unintuitive results arguably arise from properties of the underlying logic. This is illustrated in (2.5) and (2.4) where, in one fashion or another, one does not want to apply the contrapositive of a default; rather a default is to be applied in a "forward" fashion only. Under this second interpretation a default is regarded more as an (object-level) *rule*, whose properties would be closer to those of a rule of inference. Hence, given a conditional $\alpha \to \beta$, if the antecedent $\alpha$ happens to be true, we conclude $\beta$ by default. Given $\neg\beta$ we specifically do not want to conclude $\neg\alpha$.

## 3 Defaults as Rules

The general approach developed here is the same as those described in the previous section: we begin by specifying a conditional logic of defaults and subsequently provide a principled, nonmonotonic means to extend the logic to account for irrelevant properties. Our point of departure is that we informally treat defaults more like rules of inference, in that defaults are intended to be applied in a "forward" direction only.

Our interpretation, roughly, is that the antecedent of a default establishes a context in which the consequent (normally) holds, or holds all other things being equal. Thus for example, if one accepts that normally bread is eaten with the fingers, $b \rightarrow f$, then our interpretation is that $b \wedge f$ is more normal, usual, or preferable than $b \wedge \neg f$. Hence one can consider that, for default $\alpha \rightarrow \beta$, the formula $\alpha$ established a context, and in this context it is the case that $\beta$ is more normal (typical, etc.) than $\neg\beta$. We express this semantically by introducing a binary relation of relative normality $<_w$ between propositions; formula $\alpha \rightarrow \beta$ is true in a model $M$ at world $w$ if

$$\|\alpha\|_w \cap \|\neg\beta\|_w <_w \|\alpha\|_w \cap \|\beta\|_w, \tag{2}$$

that is, the proposition (see the next section) $\|\alpha\|_w \cap \|\beta\|_w$ is more normal (typical, etc.) than $\|\alpha\|_w \cap \|\neg\beta\|_w$ at world $w$. It seems reasonable that this binary relation of relative normality $<$ be asymmetric and transitive, and so we generally assume that these conditions hold.

We note that the form of (2) has appeared regularly in the literature, going back at least to [Lewis, 1973].[4] The difference is that usually the interpretation of (2) is along the lines of "the *least* worlds where $\alpha \wedge \neg\beta$ is true are less normal than the least $\alpha \wedge \beta$ worlds". Thus for example in [Lewis, 1973, pp. 54-56] $P \preceq_i Q$ is used to express that "the proposition $P$ is at least as possible, at the world $i$, as the proposition $Q$." However, as the exposition makes clear, what this notation really means is that *the least $P$-worlds are at least as possible, at the world $i$, as the least $Q$-worlds.*

Filling in the (formal) details yields a weak logic of conditionals **C**, significantly weaker than the so-called "conservative core", in which weak conditionals of the form $\alpha \rightarrow \beta$ can be interpreted. The operator $\rightarrow$ is a binary modal operator defined not in terms of accessibility among possible worlds, but rather directly in terms of pairs of propositions. For the sake of increased expressibility, it is convenient to also introduce a notion of necessity, expressed by $\Box\alpha$ for "$\alpha$ is necessarily true" or semantically, "$\alpha$ is true at all worlds considered possible". Our notion of necessity is given a physical interpretation (as opposed to, say, an epistemic interpretation). Thus we might use $\Box(k \supset c)$ to express propositionally that a knife $k$ is necessarily a piece of cutlery $c$.

Since the base logic is very weak, we also consider various strengthenings of the logic, and in the end suggest a preferred logic, that we call **C+**, for expressing statements of normality. However, notably all of these strengthenings are still weaker than the "core" set of defaults. However we show that these logics have desirable properties; as well, the undesirable inference illustrated in (2.6) is not obtained.

Moreover, it proves to be the case that nonmonotonic reasoning is definable in a very simple and straightforward manner. Consider again our example that one normally eats bread with the fingers, $b \rightarrow f$. One would also want to be able to incorporate irrelevant properties, when reasonable. Thus it would seem that barring information to the contrary, one should (nonmonotonically) accept that normally whole-wheat bread is eaten with the fingers, $b \wedge w \rightarrow f$. Semantically this would mean, that given $\|b \wedge \neg f\|_w <_w \|b \wedge f\|_w$ that one would like to extend a model to have $\|b \wedge \neg f\|_w \cap \|w\|_w <_w \|b \wedge f\|_w \cap \|w\|_w$ and so $\|b \wedge w \wedge \neg f\|_w <_w \|b \wedge w \wedge f\|_w$. But how to do this, at least in broad outline, is straightforward: Basically, the (semantic) relation $X <_w Y$ asserts that in the "context" (set of possible worlds) $X \cup Y$, partitioned by $X$,$Y$, we have that $Y$ is more normal than $X$. Our nonmonotonic assumption is that this obtains in all "feasible" subcontexts (where, of course 'feasible' needs to be defined). That is, for proposition $Z$, unless there is reason to conclude otherwise, we assert that $Z \cap X <_w Z \cap Y$. The next section develops the formal details of the family of weak conditional logics for defaults, while the following section addresses nonmonotonic reasoning in this family of logics.

## 4 The Approach

### 4.1 Formal Preliminaries

We assume some familiarity with modal logics. $\mathcal{L}_{PC}$ is the language of classical propositional logic defined, for simplicity, over a finite alphabet $\mathbf{P} = \{a, b, c, \dots\}$ of *propositional letters* or *atomic propositions*. Formulas of $\mathcal{L}_{PC}$ are constructed from the logical symbols $\neg$, $\vee$, $\wedge$, $\supset$, and $\equiv$ in the standard manner. The symbol $\top$ is taken to be some propositional tautology, and $\bot$ is defined as $\neg\top$. Our language for expressing weak conditionals, $\mathcal{L}$, is $\mathcal{L}_{PC}$ extended with the binary operator $\rightarrow$ and the unary operator $\Box$. The operator $\rightarrow$ is the *weak conditional*, in contrast to the material conditional $\supset$; the operator $\Box$ expresses necessity. For convenience, arguments of both $\rightarrow$ and $\Box$ are members of $\mathcal{L}_{PC}$; that is, we do not allow nested occurrences of $\rightarrow$ nor $\Box$. As is usual, we will use $\Diamond$ to abbreviate $\neg\Box\neg$.

Formulas are denoted by the Greek letters $\alpha$, $\beta$, $\alpha_1$, …and sets of formulas by upper case Greek letters $\Gamma$, $\Delta$, $\Gamma_1$, …. The symbol $\vdash$, possibly subscripted with the name of a system, is used to indicate derivation of a formula from a set of formulas.

The semantics is based on the notion of a *possible world*, where a possible world can be thought of a complete, consistent description of how the world could conceivably be. Each world $w$ will have associated with it a set of possible worlds $N(w)$ where $w' \in N(w)$ indicates that according to $w$, $w'$ is a possible world. Every formula will be either true or false at a world $w$ in a

[4] We use $<$ in the opposite sense of Lewis.

model $M$. That $\alpha$ is true at world $w$ in model $M$ will be written $M, w \models \alpha$. If $\alpha$ is true at every world in every model, then $\alpha$ is *valid*, written $\models \alpha$. Given a world $w$, we identify the proposition expressed by a sentence $\alpha$ at $w$ with the set of possible worlds in which $\alpha$ is true, denoted $\|\alpha\|_w$, that is,

$$\|\alpha\|_w = \{w' \in N(w) \mid \; M, w \models \alpha\}.$$

Propositions (viz. sets of possible worlds) are also denoted by the upper case letters $X_w$, $Y_w$, .... Most often we will drop the subscript $w$ since the world in question will be understood unambiguously.

## 4.2 The Base Logic

Unlike the approaches described in Section 2, we do not employ a Kripke structure on possible worlds for the interpretation of the conditional $\rightarrow$. Rather, each world in a model is associated with a binary notion of *relative normality*, denoted $<$, between sets of possible worlds, or propositions. Sentences are interpreted with respect to a model, as follows.

**Definition 4.1** *A* comparative conditional model *is a tuple* $M = \langle W, N, <, P\rangle$ *where:*

1. *$W$ is a set (of states or possible worlds);*
2. *$N : W \mapsto 2^W \setminus \emptyset$.*
3. *$< \subseteq W \times 2^W \times 2^W$ with properties described below;*
4. *$P : \mathbf{P} \mapsto 2^W$.*

$P$ maps atomic sentences onto sets of worlds, being those worlds at which the sentence is true. For $w \in W$, $N(w)$ gives the set of those worlds considered possible at $w$. We require that $w \in N(w)$. The relation $<$ associates with each world $w \in W$ a binary notion of *relative normality* between propositions; we write $X <_w Y$ to assert that, according to world $w$, proposition $Y$ is more normal than $X$. We will require that $X, Y \subseteq N(w)$ and will consider just the case where $X \cap Y = \emptyset$. That is, given a partition $\{X, Y\}$ of a subset of the possible worlds, the relation $X <_w Y$ asserts that $Y$ is more normal (unexceptional, etc.) than $X$ at world $w$. We assume that $<$ is a strict partial ordering on its last two arguments, that is for $w \in W$, $<_w$ is asymmetric and transitive. As well, we will assume that the incoherent proposition is maximally abnormal:

$$\text{If} \quad X \neq \emptyset \quad \text{then} \quad \emptyset <_w X. \tag{3}$$

Truth of a formula at a world in a model is as for propositional logic, with additions for $\Box$ and $\rightarrow$:

**Definition 4.2**

1. $M, w \models p$ *for* $p \in \mathbf{P}$ *iff* $w \in P(p)$.
2. $M, w \models \alpha \wedge \beta$ *iff* $M, w \models \alpha$ *and* $M, w \models \beta$.
3. $M, w \models \neg\alpha$ *iff* $M, w \not\models \alpha$.
4. $M, w \models \Box\alpha$ *iff* $\|\alpha\|_w = N(w)$.
5. $M, w \models \alpha \rightarrow \beta$ *iff* $\|\alpha\|_w \cap \|\neg\beta\|_w <_w \|\alpha\|_w \cap \|\beta\|_w$.

Thus $\alpha \rightarrow \beta$ is true just if the proposition expressed by $\alpha \wedge \beta$ is more normal than that expressed by $\alpha \wedge \neg\beta$.

Consider the logic closed under classical propositional logic along with the following rules of inference and axioms:

**Nec:** From $\vdash \alpha$ infer $\vdash \Box\alpha$.

**K:** $\vdash \Box(\alpha \supset \beta) \supset (\Box\alpha \supset \Box\beta)$.

**T:** $\vdash \Box\alpha \supset \alpha$.

**CEA:** $\vdash \Box(\alpha \equiv \alpha') \supset ((\alpha \rightarrow \beta) \equiv (\alpha' \rightarrow \beta))$.

**CECA:** $\vdash \Box(\alpha \supset (\beta \equiv \beta')) \supset ((\alpha \rightarrow \beta) \equiv (\alpha \rightarrow \beta'))$.

**RR:** $\vdash \Diamond\alpha \supset (\alpha \rightarrow \alpha)$.

**NA:** $\neg(\bot \rightarrow \alpha)$

**CEM:** $(\alpha \rightarrow \beta) \supset \neg(\alpha \rightarrow \neg\beta)$

**Trans:** $\vdash \Box(\neg(\alpha_1 \wedge \alpha_2) \wedge \neg(\alpha_1 \wedge \alpha_3) \wedge \neg(\alpha_2 \wedge \alpha_3)) \supset ([(\alpha_1 \vee \alpha_2 \rightarrow \alpha_2) \wedge (\alpha_2 \vee \alpha_3 \rightarrow \alpha_3)] \supset (\alpha_1 \vee \alpha_3 \rightarrow \alpha_3))$

We call the smallest logic based on the above axiomatisation **C**. **Nec**, **K**, and **T** characterise $\Box$. **CEA** (Conditional Equivalent Antecedents) gives substitution of necessary equivalents in the antecedent of a conditional. **CECA** (Conditional Equivalent Consequents, given Antecedents) asserts the same thing with respect to consequents, but is somewhat more general, in that the consequents need be equivalent just in the "context" given by the antecedent. **RR** is restricted reflexivity; here as with other conditionals we disallow the incoherent proposition $\bot$ to be the antecedent of a true conditional. **NA** (Nihil ex Absurdo) asserts that the incoherent proposition never normally implies anything. [Benferhat *et al.*, 1992] expresses this axiom nicely, that "while $\bot$ (classically) entails anything, it should preferentially entail nothing". **CEM** is the excluded middle for a weak conditional; in the semantics this is reflected by asymmetry of $<$. Similarly **Trans** reflects transitivity of $<$ in the semantics. We obtain the following basic results:

**Theorem 4.1**

1. *From* $\vdash \alpha \equiv \alpha'$ *infer* $\vdash (\alpha \rightarrow \beta) \equiv (\alpha' \rightarrow \beta)$
2. *From* $\vdash \beta \equiv \beta'$ *infer* $\vdash (\alpha \rightarrow \beta) \equiv (\alpha \rightarrow \beta')$
3. $\vdash (\alpha \rightarrow \beta) \supset (\alpha \rightarrow (\alpha \wedge \beta))$
4. $\vdash (\alpha \rightarrow \beta) \supset (\alpha \rightarrow (\alpha \supset \beta))$
5. $\vdash \Diamond\alpha \supset (\Box(\alpha \supset \beta) \supset (\alpha \rightarrow \beta))$.
6. $\vdash (\Diamond\alpha \wedge \Box\beta) \supset (\alpha \rightarrow \beta)$
7. $\vdash \neg(\alpha \rightarrow \bot)$

The first two results express substitution of logical equivalents in the antecedent and consequent of a conditional. These rules have been called **RCEA** and **RCEC** in the conditional logic literature. The next two results effectively express the range of equivalent forms a conditional may take on with respect to the consequent. The following two results, Parts 4 and 5, connect the modalities $\Box$ and $\rightarrow$. The former is analogous to **RCE** in the conditional logic literature, while the latter is slightly weaker than **MOD**. The last result asserts that no proposition normally implies the incoherent proposition.

Of those axioms in the presumed "core" logic (Section 2), **RCM**, **RT**, **ASC**, **CC**, and **CA** are not valid in **C**. Nonetheless, despite its (monotonic) inferential weakness, the logic allows a rich set of nonmonotonic inferences, as covered in the next section. However, first we explore properties of the (monotonic) logic.

**Properties of the Logic**

Soundness of the logic is shown by a straightforward inductive argument. For the completeness proof, it is of interest to first consider the weakest logic compatible with the semantic framework given in Definition 4.1. Completeness is given with respect to the weakest realistic semantic framework; the corresponding weakened axiomatic system is called **C**−. Then the roles of nontriviality, asymmetry and transitivity in the semantics is clearly reflected in the axioms that are added to **C**−, yielding the system **C**.

**Definition 4.3** *A* weak comparative conditional model *is a comparative conditional model (Definition 4.1), $M = \langle W, N, <, P\rangle$ except that we define $<$ simply by $< \subseteq W \times 2^W \times 2^W$.*

Consider the logic over $\mathcal{L}$ closed under classical propositional logic and **Nec**, **K**, **T**, along with the following axioms:

**CEA:** $\vdash \Box(\alpha \equiv \alpha') \supset ((\alpha \to \beta) \equiv (\alpha' \to \beta))$.

**CECA:** $\vdash \Box(\alpha \supset (\beta \equiv \beta')) \supset ((\alpha \to \beta) \equiv (\alpha \to \beta'))$.

The smallest logic based on the above axiomatisation is called **C**−. It is easily shown that **C**− is sound with respect to weak comparative conditional models.

Completeness is demonstrated by constructing a *canonical model* [Chellas, 1980; Hughes and Cresswell, 1996] for the logic **C**−, that is, a model such that every non-theorem of **C**− is false at some world in the model.

We obtain the characterization result:

**Theorem 4.2** *$\alpha$ is valid in the class of weak comparative conditional models iff $\vdash \alpha$ in* **C**−.

Given this result, we can next consider the addition of properties to the logic that will strengthen **C**− to our "official" base logic **C**. If $<_w$ is to be interpreted as capturing a notion of normality between propositions, then there are arguably four properties that are necessary, in that if any were omitted, then $<_w$ would be arguably too weak to capture this notion of normality. These properties are as follows:

1. Min $\emptyset_1$: $\emptyset <_w X$ for every $\emptyset \neq X \subseteq N(w)$.
2. Min $\emptyset_2$: $X \not<_w \emptyset$ for every $X \subseteq N(w)$.
3. Asymmetry: If $X <_w Y$ then $Y \not<_w X$.
4. Transitivity: If $X <_w Y$, $Y <_w Z$ then $X <_w Z$ provided $X$, $Y$, $Z$ are pairwise disjoint.

We obtain the following correspondence between these semantic conditions and their corresponding axioms:

**Theorem 4.3**

**C**− + *min $\emptyset_1$ (min $\emptyset_2$, asymmetry, transitivity) is complete with respect to the class of weak comparative conditional models closed under* **RR** *(***NA***,* **CEM***,* **Trans***).*

**Corollary 4.1** **C** *is complete with respect to the class of comparative conditional models.*

### 4.3 Extensions to the Logic

In the logic **C** most properties of the relation $<_w$ stem from its being a strict partial order (viz. asymmetric and transitive), along with the fact that $\emptyset$ is the minimum in $<_w$. In this subsection we look at strengthening $<_w$ by considering properties that seem reasonable for a notion of normality. Consider the following:

1. Continues Down:
   If $X <_w Y$ then $X \setminus Z <_w Y$.
2. Continues Up:
   If $X <_w Y$ then $X <_w Y \cup Z$ provided $X \cap Z = \emptyset$.
3. Restricted Continues Down/Up:
   If $X <_w Y$ then $X \setminus Z <_w Y \cup (X \cap Z)$.
4. Continues Down/Up:
   If $X <_w Y$ then $X \setminus Z <_w Y \cup Z$.
5. Weak Disjoint Union:
   If $X_1 <_w Y_1$, $X_2 <_w Y_2$, $Y_1 \cap Y_2 = \emptyset$ and $(X_1 \cup X_2) \cap (Y_1 \cup Y_2) = \emptyset$ then $X_1 \cup X_2 <_w Y_1 \cup Y_2$.

For Continues Down, if $X$ is less normal than $Y$, then a stronger proposition than $X$ (viz. $X \setminus Z$) is also less normal than $Y$. That is, removing worlds from a proposition serves to strengthen it, thereby making it "less normal". Continues Up is a dual: if $X$ is less normal than $Y$, then a weaker proposition than $Y$ (viz. $Y \cup Z$) is also more normal than $X$. That is, adding worlds to a proposition serves to make it "more normal". For Restricted Continues Down/Up, if $X < Y$, then "part" of $X$ can be shifted to $Y$; hence $X$ is strengthened and $Y$ weakened by the same set of worlds. Continues Down/Up combines Continues Down and Continues Up, as well as generalises Restricted Continues Down/Up. Weak Disjoint Union allows for the combination of two independent instances of $<_w$, provided that the result yields a relation in which the arguments are disjoint.

Interestingly, Continues Down, Continues Up, and their combination have appeared in the belief revision literature. Our relation $<$ is what [Alchourrón and Makinson, 1985] call a *(transitive) hierarchy*;[5] while $<$ with Continues Down/Up is a *regular* hierarchy. Their interpretation of $<$ echoes ours for $X < Y$, that "$X$ is less secure or reliable or plausible ... than $Y$" [Gärdenfors and Rott, 1995, p. 75].

Consider next the following formulas:

[5] In [Alchourrón and Makinson, 1985], $<$ is a binary relation on deductively-closed sets of sentences which, in the finite case, serves as well as sets of worlds for expressing propositions.

**WSA:** $\Box(\beta \supset \gamma) \supset ((\alpha \to \beta) \supset (\alpha \wedge \gamma \to \beta))$

**CW:** $(\alpha \wedge \beta \to \gamma) \supset (\alpha \to (\beta \supset \gamma))$

**CM:** $\Box(\beta \supset \gamma) \supset ((\alpha \to \beta) \supset (\alpha \to \gamma))$

**WCA:** $(\alpha \to \beta) \supset ((\alpha \vee \gamma) \to (\beta \vee \gamma))$.

**D:** $((\alpha \wedge \beta \to \gamma) \wedge (\alpha \wedge \neg\beta \to \gamma)) \supset (\alpha \to \gamma)$.

**WSA** (Weak Strengthening of the Antecedent) is a stronger version of the easy result in **C**: $\Box\gamma \supset ((\alpha \to \beta) \supset (\alpha \wedge \gamma \to \beta))$. **CW** (Conditional Weakening, or "Conditionalisation" in nonmonotonic consequence relations) gives a conditional version of one half of the deduction theorem. The formula **CM** allows weakening of the consequent of a conditional. In conditional approaches it more often appears in a weaker form, as a rule of inference, where it is called **RCM** or (as a nonmonotonic inference relation) Right Weakening. As well, this formula can be seen as allowing a form of modus ponens in the consequent of a conditional. Combining **WSA** and **CW** yields the formula **WCA** (Weak **CA**), which allows incorporation of information uniformly in the antecedent and consequent of a conditional. **D** supplies a certain "reasoning by cases" for the conditional.

We obtain the following correspondence between semantic conditions and the axiomatisation:

**Theorem 4.4**

**C** + *Continues Down (Continues Up, Restricted Continues Down/Up, Continues Down/Up, Weak Disjoint Union) is complete with respect to the class of comparative conditional models closed under* **WSA** *(***CW***,* **CM***,* **WCA***,* **D***).*

For ease of discussion, let **C+** be the logic:

**C + WSA + WCA + WDJ = C + WSA + CW + CM + WDJ**.

For purposes of discussion[6] we will take **C+** as our "official" logic for a rule-based interpretation of defaults and statements of normality.

We obtain the following easy results:

**Theorem 4.5**

1. $\vdash_{\mathbf{C+}} (\alpha \to (\beta \wedge \gamma)) \supset (\alpha \to \beta)$
2. *From* $\vdash \beta \supset \gamma$ *infer* $\vdash_{\mathbf{C+}} (\alpha \to \beta) \supset (\alpha \to \gamma)$

The first (called **CM**) is a converse to **CC**/And. The second, (**RCM** or Right Weakening) is a common condition for conditional approaches; it is a weaker version of our **CM**.

The following formulas are not derivable in **C+**:

1. $(\alpha \to (\beta \supset \gamma)) \supset ((\alpha \to \beta) \supset (\alpha \to \gamma))$
2. $((\alpha \to \gamma) \wedge (\beta \to \gamma)) \supset (\alpha \vee \beta \to \gamma)$

The first of these is called **CK** in the conditional logic literature; **MPC** in that of nonmonotonic inference relations. It is the natural strengthening of **CM** to modus ponens in the consequence of a conditional. The second is called **CA** or **OR**. It is a natural strengthening of Theorem 4.5.1. As mentioned, there are authors (e.g. [Neufeld, 1989]) who see this formula as problematic. As well, various other characterizing formulas in preferential systems are not theorems here, specifically **CC**/And, **RT**/Cut, and **ASC**/Cautious Monotony. (The lack of **CC**/And means that we do not have what [Gärdenfors and Makinson, 1994] call an inference system.)

We defer further analysis and discussion to the full paper, in which we argue that for our purposes these formulas are not necessarily desirable. In Section 6 we briefly consider further extensions to the logic.

[6] Which is to say, the point is open to debate.

## 5 Considerations on Nonmonotonic Reasoning

We claimed at the outset that the logic **C** and its strengthenings would allow a simple approach to nonmonotonic inference, having just the "right" properties for a rule-based interpretation of a conditional. For these logics, the central idea is that, given a partition $\{X, Y\}$ of a context $X \cup Y \subseteq W$, the relation $X <_w Y$ asserts that $Y$ is more normal (unexceptional, etc.) than $X$. To obtain nonmonotonic inference, we simply assume that this relation holds in any subcontext, that is $X \cap Z <_w Y \cap Z$, wherever "reasonable".

Informally this notion of "reasonable" is straightforward to specify:

> If we have $X <_w Y$ then assert $X \cap Z <_w Y \cap Z$ just when, for every $X'$, $Y'$ where
> $X \cap Z \subseteq X' \subseteq X$ and $Y \cap Z \subseteq Y' \subseteq Y$
> we don't have $Y' <_w X'$.

As well we have the constraint that $Y \cap Z \neq \emptyset$. More formally, we have the following:

**Definition 5.1** *Let* $M = \langle W, N, <, P\rangle$ *be a comparative conditional model in* **C**.
*Define* $M^* = \langle W, N, <^*, P\rangle$*, an* augmentation *of* $M$*, by:*

1. $M^*$ *is a comparative conditional model, and*
2. $X <^*_w Y$ *iff* $Y \neq \emptyset$ *and there are* $X' \supseteq X$*,* $Y' \supseteq Y$ *such that*
   (a) $X' <_w Y'$ *and*
   (b) *for every* $X''$*,* $Y''$ *where* $X \subseteq X''$*,* $Y \subseteq Y''$ *and* $Y'' <_w X''$ *we have:* $X' \subseteq X''$*,* $Y' \subseteq Y''$.

It can be noted that in Definition 5.1, if we have Continues Down, then we don't need to bother with $X'$ and $X''$, since monotonically we get that if $X' < Y'$ then $X < Y'$ for any $X \subseteq X'$. As well, Definition 5.1 in combination with Continues Up may perhaps give too many relations: From $X < Y$ we have $X < Y \cup Z$ by Continues Up and then, all other things being equal we nonmonotonically conclude $X < Z$. Thus from $X < Y$ and arbitrary (but in accordance with the conditions of the definition) $Z$, we obtain that $X < Z$. Assuming that this is a problem, then there are two ways in which this difficulty can be resolved. First, one can decide that

Continues Up (and so **CW**) is too strong for our logic of conditionals. Or, second, the Definition 5.1 can be restricted to apply to certain "minimal" sets of worlds.

We define $\models^*$ as validity in the class of augmented comparative conditional models; that is $\models^* \alpha$ iff $\alpha$ is true at every world in every augmented comparative conditional model. Nonmonotonic inference is defined as follows:

**Definition 5.2** *Let* $\Gamma \subseteq \{\alpha \to \beta \mid \alpha, \beta \in \mathcal{L}_{PC}\} \cup \{\Box\alpha \mid \alpha \in \mathcal{L}_{PC}\}$.

*Define:* $\alpha \mid\!\sim_\Gamma \beta$ *iff* $\models^* \Gamma \supset (\alpha \to \beta)$.

We say that $\beta$ is a nonmonotonic inference from $\alpha$ with respect to $\Gamma$, or just $\beta$ is a nonmonotonic inference from $\alpha$ if the set $\Gamma$ is clear from the context of discussion.

We illustrate nonmonotonic inference first by a familiar example:

$$b \to f, \tag{4}$$
$$b \to w, \tag{5}$$
$$\Box(p \supset b), \tag{6}$$
$$p \to \neg f. \tag{7}$$

Thus birds fly and have wings, and penguins are (necessarily) birds that do not fly. We obtain the following:

$$\begin{array}{ll} b \wedge w \mid\!\sim f, & p \wedge w \mid\!\sim \neg f, \\ b \wedge \neg w \mid\!\sim f, & p \wedge b \mid\!\sim \neg f, \\ b \wedge \neg p \mid\!\sim f, & p \wedge b \wedge w \mid\!\sim \neg f. \end{array}$$

We also obtain $b \wedge x \wedge y \wedge z \mid\!\sim w$ for $x \in \{\top, g, \neg g\}, y \in \{\top, p, \neg p\}, z \in \{\top, f, \neg f\}$. Thus green ($g$) birds have wings, as do non-green flying penguins. As well $p \mid\!\sim w$, and so penguins inherit the property of having wings by virtue of necessarily being birds. Note that if we replaced (6) by $p \to b$, we would no longer obtain $p \mid\!\sim w$; however we would obtain the weaker $b \wedge p \mid\!\sim w$. We justify this by noting that a normality conditional $\alpha \to \beta$ does not imply a strict specificity relation between $\alpha$ and $\beta$ whereas $\Box(\alpha \supset \beta)$ does.

The next example further illustrates reasoning in the presence of exceptions.

$$q \to p, \quad r \to \neg p, \quad q \to g \tag{8}$$

So Quakers are pacifists while Republicans are not, and Quakers are generous. We obtain $q \wedge \neg r \mid\!\sim p$ and $q \wedge r \mid\!\sim g$. Thus in the last case, while Quakers that are Republican are, informally, exceptional Quakers, they are nonetheless still generous by default.

We do not obtain the undesirable inference $\neg o \mid\!\sim \neg a$, noted in Example 2.3 in Section 2. Further, for Example 2.4 we do not obtain $f \mid\!\sim \neg u$. Last, we note that while we obtain full incorporation of irrelevant properties, we do not obtain full default transitivity. Thus

$$a \to b, \quad b \to c \tag{9}$$

does not yield $a \mid\!\sim c$ (nor, incidentally, do we obtain $\neg b \mid\!\sim \neg a$). However we do get $a \wedge b \mid\!\sim c$. If we replaced $a \to b$ with $\Box(a \supset b)$ we would get $a \mid\!\sim c$. If we replaced $b \to c$ with $\Box(b \supset c)$ we would again get $a \mid\!\sim c$, in **C** (in fact we could derive $a \to c$ in those logics containing **CM**, as given in Theorem 4.3).

# 6 Discussion

We could go on and add other conditions in the semantics. Space considerations dictate against a lengthy discussion, but two conditions are worth noting here:

**Disjoint Union:** If $(X \cup Y) \cap Z = \emptyset$ then: $X <_w Y$ iff $X \cup Z <_w Y \cup Z$.

**Connectivity:** For $X, Y \subseteq W$, either $Y <_w X$ or $X <_w Y$.

Disjoint union has appeared frequently in the literature, for example [Savage, 1972; Fine, 1973; Dubois *et al.*, 1994]. The addition of disjoint union requires that the notion of a model be altered slightly (from a relation $<$ to $\leq$); the resultant semantic framework would correspond to the basic definition of a *plausibility structure* [Friedman and Halpern, 2001]. The addition of connectivity would make $<_w$ a *qualitative probability* in the terminology of [Savage, 1972].

# 7 Conclusion

We have argued that there are two interpretations of a default conditional: as a weak (typically material) implication, or as something akin to a rule of inference. The former interpretation is explicit in, for example, circumscriptive abnormality theories, and implicit in an approach such as conditional entailment. It is clear that there are many, and varied, applications in which the first interpretation is appropriate. However we have also noted that there are various reasons to suppose that this is not the only such interpretation: First, work such as [Poole, 1991] and [Neufeld, 1989] can be viewed as arguing against principles of the "core" logic underlying this first interpretation (the former arguing against the principle **CC**/And and the latter against **CA**/Or). Second, there are examples of inferences in approaches such as rational closure or in conditional entailment that are either too weak or too strong. Last, there are emerging areas (such as causal reasoning) in which a "weak material implication" interpretation is not appropriate. While this distinction has been recognized previously, what is new here is the development of a family of logics, with a novel semantic theory and proof theory, along with a specification of nonmonotonic inference, for the "rule-based" interpretation.

All of the logics presented here are quite weak, at least compared to the "conservative core" or, equivalently, the system **P** of [Kraus *et al.*, 1990]. We argue however that such lack of inferential capability is characteristic of a "rule-based" interpretation of a conditional. Moreover it proves to be the case that nonmonotonic reasoning is defined very easily in these logics, and allows a rich set of inferences concerning the incorporation of irrelevant properties and of property inheritance.

An open question concerns how informal, commonsense defaults should be classified – whether as a defeasible classical conditional or as a rule. Certainly past work has favoured the "defeasible classical conditional" interpretation. However, a case can be made that many examples formerly interpreted as belonging to the first category are better interpreted as belonging to the "rule" category. Consider Lewis' approach to counterfactuals [Lewis, 1973] in which the following example, concerning a past party, is given: "If John had gone it would have been a good party" and "If John and Mary had gone it would have not been a good party". From this we deduce that "if John had gone, Mary would not have gone". This, to most readers, is a strange result: John's going and Mary's going are (presumably) independent events. Arguably this result ought not to obtain, and so perhaps counterfactuals, as previously modelled by Lewis' sphere semantics, may be better interpreted via the "rule" interpretation.

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