Real-Time Problem-Solving with Contract Algorithms

Correction of the Proofs of Theorems 2 and 4

**Theorem 2** The minimal acceleration ratio needed to construct an interruptible algorithm from a given contract algorithm is \( r = 4 \).

**Proof:** From Lemma 1 we know that for any sequence of contracts, \( X = (x_1, x_2, \ldots) \), \( r \) must satisfy:

\[
\forall i \geq 1 : \quad Q_A\left(\frac{x_1 + x_2 + \cdots + x_{i+1}}{r}\right) \leq Q_A(x_i)
\]

From the strict monotonicity of \( Q_A \) we get:

\[
\forall i \geq 1 : \quad \sum_{j=1}^{i+1} x_j \leq r x_i
\]

Setting:

\[
g_0 = 0 \quad \text{and} \quad g_i = \sum_{j=1}^{i} x_j, \quad i \geq 1,
\]

we can write the previous equation as:

\[
\forall i \geq 1 : \quad g_{i+1} \leq r (g_i - g_{i-1}). \quad (4)
\]

We know that the sequence \( (g_i)_{i \geq 1} \) is an increasing sequence of positive numbers, so \( \rho \), defined as follows:

\[
\rho = \inf\{\frac{g_2}{g_1}, \frac{g_3}{g_2}, \ldots, \frac{g_{i+1}}{g_i}, \ldots\},
\]

satisfies \( \rho \geq 1 \). From Equation (4) we obtain

\[
\forall i \geq 1 : \quad g_i \geq g_{i-1} + \frac{g_{i+1}}{r},
\]

and thus

\[
\forall i \geq 2 : \quad \frac{g_i}{g_{i-1}} \geq 1 + \frac{g_{i+1}}{r g_{i-1}} \geq 1 + \frac{\rho^2}{r}.
\]

Finally, we deduce that:

\[
\rho \geq 1 + \frac{\rho^2}{r}
\]

(so that \( \rho > 1 \)) or that:

\[
r \geq \frac{\rho^2}{\rho - 1} \geq 4,
\]

as the function \( \rho \rightarrow \frac{\rho^2}{\rho - 1} \) reaches a minimum of 4 on the interval \((1, +\infty)\) for \( \rho = 2 \). \( \square \)

**Theorem 4** The minimal acceleration ratio needed to construct an interruptible algorithm to solve \( m \) problem instances with a given contract algorithm is \( r = (\frac{m+1}{m})^{m+1} \).

**Proof:** For any sequence of contracts, \( X = (x_k)_{k \geq 1} \), \( r \) must satisfy:

\[
\forall i \geq 1 : \quad Q_A\left(\frac{x_1 + x_2 + \cdots + x_{i+m}}{mr}\right) \leq Q_A(x_i)
\]

From the strict monotonicity of \( Q_A \) we get:

\[
\forall i \geq 1 : \quad \sum_{j=1}^{i+m} x_j \leq mr x_i
\]

Setting:

\[
g_0 = 0 \quad \text{and} \quad g_i = \sum_{j=1}^{i} x_j, \quad i \geq 1,
\]

we can write the previous equation as:

\[
\forall i \geq 1 : \quad g_{i+m} \leq mr (g_i - g_{i-1}).
\]

or

\[
\forall i \geq 1 : \quad g_i \geq g_{i-1} + \frac{g_{i+m}}{mr}, \quad (9)
\]

We know that the sequence \( (g_i)_{i \geq 1} \) is an increasing sequence of positive numbers, so \( \rho \), defined as follows:

\[
\rho = \inf\{\frac{g_2}{g_1}, \frac{g_3}{g_2}, \ldots, \frac{g_{i+1}}{g_i}, \ldots\},
\]

satisfies \( \rho \geq 1 \). From Equation (9) we obtain

\[
\forall i \geq 2 : \quad \frac{g_i}{g_{i-1}} \geq 1 + \frac{g_{i+m}}{mr g_{i-1}} \geq 1 + \frac{\rho^{m+1}}{mr}.
\]

Finally, we deduce that:

\[
\rho \geq 1 + \frac{\rho^{m+1}}{mr}
\]

(so that \( \rho > 1 \)) or that:

\[
r \geq \frac{\rho^{m+1}}{m(\rho - 1)}.
\]

The function \( \rho \rightarrow \frac{\rho^{m+1}}{\rho - 1} \) reaches its minimum on the interval \((1, +\infty)\) when \( \rho = \frac{m+1}{m} \), therefore \( r \geq (\frac{m+1}{m})^{m+1} \).

The ratio \((\frac{m+1}{m})^{m+1}\) can be obtained by a sequence of contracts defined by a geometric series with run-times being multiplied by a factor of \( \frac{m+1}{m} \). Thus the best possible acceleration ratio is \( r = (\frac{m+1}{m})^{m+1} \). \( \square \)

*We are grateful to Reshef Meir for pointing out the error in the original proofs.*